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# A Class of Improper Boundary Value Problems with "Damped" Cauchy Data

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## I. INTRODUCTION

In this paper we shall study strong solutions to the Cauchy problem for a certain class of non-linear partial differential equations. In general the problems we consider are improper in the sense of Hadamard[1]. That is the solutions do not depend continuously on the initial data in any reasonably defined norm. The general class of problems treated includes as special cases such classical instances of improperness as the Cauchy problem for elliptic equations and timelike initial value problems for hyperbolic equations.

There are at least two ways of approaching improper problems with the aim of obtaining solutions which do possess satisfactory continuous dependence properties. One is to seek solutions only within a restricted class of functions (e.g. functions with a fixed finite bound in a suitably chosen norm). This approach has been extensively investigated by F. John and quite complete result have been obtained[2].

The second method which has been employed by Miranker[3] and the present author[4] is based on the restriction of the class of allowable initial and boundary data. In the latter work the Cauchy data is assumed to be band-limited, that is to possess Fourier transforms with compact support in all variables. This assumption is motivated by consideration of the limitations of measuring systems which provide the initial or boundary data in many actual physical problems. Under such a stringent assumption

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it is not very surprising that very strong results can be obtained. In fact we have shown in [4] for a class of nonlinear problems analogous to the ones treated in this paper that the resulting continuous dependence of the solution is of the Lipschitz type.

Several questions arise naturally in connection with the method of restricting the class of allowable data: (1) when the operator involved in the differential equation is of a specified type (e.g. hyperbolic), can the assumption of bandlimitedness be limited to a subset of the independent variables and strong continuous dependence of the solution still be demonstrated; (2) more generally, when the type of the operator is unspecified, can bandlimitedness be replaced by a weaker assumption on the behavior of the initial data without losing the Lipschitz continuous dependence of the solution.

In [5] we have given a partial answer to the first question by showing that for timelike initial value problems for hyperbolic equations it is sufficient to assume that the initial data is bandlimited only with respect to the space variables. The resulting continuous dependence of the solution is still of the Lipschitz type.

In [6] we continued our study of timelike hyperbolic problems by combining the results of [5] with a relaxation of the assumptio of bandlimitedness. Instead it was assumed that the Fourier transforms of the initial data with respect to the space variables possessed a certain exponential growth at infinity. Moreover, we showed in [6] that this specified growth was the best requirement of this type that could be placed on the initial data and still

obtain solutions with Lipschitz continuous dependence.

In this paper we shall treat a class of nonlinear (more precisely, quasi-linear) Cauchy problems which are similar to those discussed in [4], and in which the operator type is unspecified. We replace the bandlimited assumption of [4] by a growth condition on the Fourier transforms of the initial data (with respect to all variables) analogous to that employed in [6]. Our present results together with those of [6] provide a fairly general answer to the second questione raised above.

The writer is indebted to Professors F. John and J. Moser for several helpful discussions and suggestions.

## II. FORMULATION OF THE PROBLEM

Let k be an integer, $\alpha$ a positive real number and $\omega$ a real variable. We define a function $\phi_k^\alpha(\omega)$ by

$$(1) \qquad \phi_k^\alpha(\omega) = \begin{cases} e^{\alpha|\omega|}\omega^k & , |\omega| > 1 \\ e^\alpha & , |\omega| \leq 1 \end{cases} .$$

Note that $\phi_k^\alpha(\omega)$ is continuous for $-\infty < \omega < \infty$. We say that a real valued function g of a real variable x belongs to the class $E_k^\alpha(x)$ if

(a) $g \in L_2(-\infty,\infty)$

(b) The Fourier transform $G(\omega)$ of $g(x)$ satisfies

$$G(\omega)\,\phi_k^\alpha(\omega) \in L_2(-\infty,\infty) .$$

From (b) we have

(b') There exists an $L_2$ function $g_k^\alpha(x)$ such that

$$G(\omega)\,\phi_k^\alpha(\omega) = G_k^\alpha(\omega)$$

where $G_k^\alpha(\omega)$ is the Fourier transform of $g_k^\alpha(x)$. Explicitly,

$$G_k^\alpha(\omega) = \begin{cases} e^{\alpha|\omega|}\, G(\omega)\,\omega^k & , |\omega| > 1 \\ e^\alpha G(\omega) & , |\omega| \le 1 \end{cases} .$$

From these conditions it follows that g, when thought of as a function of the complex variable z, is analytic for $|\text{Imag } z| < \alpha$. The class of transforms of functions of $E_k^\alpha(x)$ will be denoted by $\mathcal{E}_k^\alpha(\omega)$. A real valued function $g(x,y)$ of two real variables x,y, where y is restricted to the interval $0 \le y \le h$ will be said to be of class $E_{k,m}^\alpha(x,h)$ if for every $y \in [0,h]$ $g(x,y) \in E_k^\alpha(x)$, $g(x,y) \in C^m(y)$, and if the $L_2$ norm of g with respect to x is uniformly bounded for $y \in [0,h]$ (i.e:, $||g|| \equiv \sup_{0\le y\le h} \int_{-\infty}^{\infty} |g(x,y)|^2 dx$ $\equiv \sup_{0\le y \le h} ||g(y)|| < \infty$). It is clear that $E_{k,m}^\alpha(x,h)$ (and its class of x-Fourier transforms $\mathcal{E}_{k,m}^\alpha(\omega,h)$) is closed in $L_2$, for each fixed $y \in [0,h]$.

For an arbitrary $L_1(-\infty,\infty)$ function $h(x)$ with Fourier transform $H(\omega)$, we denote by $[h(x)]_\beta$ ($\beta$ positive and real) the inverse transform of the function $H(\omega)e^{-\beta|\omega|}$. Thus $[h(x)]_\beta$ is a damped or smoothed version of $h(x)$. This operation may be thought of as a convolution in x-space

$$h(x) * d(x)$$

where the smoothing function $d(x)$ is given by

$$d(x) = \frac{2\beta}{\beta^2+x^2} .$$

Let the linear operator L be defined by

$$L = \sum_{p+q\leq m} a_{pq}(y) \frac{\partial^{p+q}}{\partial x^p \partial y^q} \quad , \tag{2}$$

where $a_{pq}(y) \in C^1(y)$ for $0 \leq y \leq h$ and $a_{0,m} \equiv 1$. Our goal is to prove the existence, uniqueness and continuous dependence of $L_2$ solutions to the equation

$$Lu = [f(u)]_\beta \tag{3}$$

in an infinite strip $R(h\ )$ defined by $-\infty < x < \infty$, $0 \leq y \leq h$, given Cauchy data

$$\frac{\partial^q u(x,0)}{\partial y^q} = v_g(x) \quad , \quad q = 0,1,\ldots,m-1 \tag{4}$$

on the x-axis satisfying $v_q(x) \in E^\alpha_{m-q}(x)$ for all q. Here $\alpha$ and $\beta$ are positive real constants (to be determined later) which depend on the coefficients $a_{pq}$, and $[f(u)]_\beta$ means that f(u) is damped as a function of x.

It will be assumed that f(u) satisfies the following conditions (which are not all independent) as a function of u:

(i) $f \in C^0(u)$

(ii) f(u) has the quadratic property in a sphere $S_\lambda$ of radius $\lambda$ (to be explicitly determined later). That is, if $||u|| \leq \lambda$, then

$$|f(u)| \leq A_\lambda u^2 \ {}^*$$

where the constant $A_\lambda$ depends only on the sphere $S_\lambda$ .

---

[*] This condition is essentially equivalent to assuming that $f(u) \in C^2(u)$ - see Ref [4].

(iii) $f(u)$ satisfies a Lipschitz condition in $S_\lambda$; i.e., if $u,u' \in S_\lambda$, then

$$|f(u) - f(u')| \leq K_\lambda \, |u-u'| \, ,$$

where $K_\lambda$ depends only on $S_\lambda$. It follows immediately from (ii) that $f(0) = 0$ and

$$\int_{-\infty}^{\infty} |f(u(x,y))|dx \leq A_\lambda \int_{-\infty}^{\infty} |u(x,y)|^2 dx = A_\lambda ||u(y)||^2 \, .$$

Thus if $u \in S_\lambda$, the Fourier transform

$$(5) \qquad F_u(\omega,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u(x,y))e^{-i\omega x}\,dx$$

exists and $[f(u)]_\beta$ may be explicitly defined by

$$(6) \qquad [f(u)]_\beta = \int_{-\infty}^{\infty} F_u(\omega,y)\, e^{-\beta|\omega|+i\omega x}\, d\omega \quad .$$

We denote the transform of $[f(u)]_\beta$ by $[F_u(\omega,y)]_\beta$. From (5) it follows that $F_u(\omega,y)$ satisfies the inequality

$$(7) \qquad |F_u(\omega,y)| \leq \frac{1}{2\pi} A_\lambda ||u(y)||^2 \quad .$$

## III AN EQUIVALENT INTEGRAL EQUATION

Our proof of the existence, uniqueness and continuous dependence of solutions to (3), (4) will depend upon the formulation of an equivalent integral equation.

First we formally write down the reduced (ordinary) differential equation associated with (3), (4)

$$(8) \qquad \mathcal{L}\, U \equiv \sum_{q=o}^{m} c_{m-q}(\omega,y) \frac{\partial^q U(\omega,y)}{\partial y^q} = [F_u(\omega,y)]_\beta \quad ,$$

where

$$(9)\qquad c_{m-q}(\omega,y) = \sum_{p=0}^{m-q} a_{pq}(y)(i\omega)^p$$

$$(10)\qquad U(\omega,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}\,dx$$

and

$$(11)\qquad \frac{\partial^q U(\omega,0)}{\partial y^q} \equiv V_q(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v_q(x)e^{-i\omega x}\,dx$$

for $q = 0,1,\ldots, m-1$. Note that $c_k(\omega,y)$ is a polynomial in $\omega$ of degree $k$.

Our program will be to use the variation of parameters method to obtain an explicit integral equation for $U$ the solution to (8), (11). To do this we first consider the solution $U_0(\omega,y)$ to the homogeneous equation

$$(12)\qquad \mathcal{L}U_0 = 0 \quad , \quad \frac{\partial^q U_0(\omega,0)}{\partial y^q} = V_q(\omega) \quad ; \ q = 0,1,\ldots,m-1.$$

Equations (12) can be written as a system and estimates of the growth of $U_0(\omega,y)$ obtained in the following way. We distinguish two cases: Case A: $|\quad| > 1$. Define the vectors $\underline{W}_0(\omega,y)$, $\underline{\tilde{V}}(\omega)$ and the matrix $\mathcal{M}(\omega,y)$ by

$$(13)\qquad \underline{W}_0(\omega,y) = \begin{bmatrix} U_0(\omega,y) \\ \frac{1}{\omega}\,\frac{\partial U_0(\omega,y)}{y} \\ \\ \frac{1}{\omega^{m-1}}\,\frac{\partial^{m-1}U_0(\omega,y)}{\partial y^{m-1}} \end{bmatrix}, \quad \underline{V}(\omega) = \begin{bmatrix} V_0(\omega) \\ \frac{1}{\omega}V_1(\omega) \\ \vdots \\ \frac{1}{\omega^{m-1}}V_{m-1}(\omega) \end{bmatrix}$$

$$(14)\quad \mathcal{M}(\omega,y) = \begin{bmatrix} 0 & 1 & 0 \cdots\cdots & 0 \\ 0 & 0 & 1 \cdots\cdots & 0 \\ \vdots & & & \\ 0 & 0 & 0 \cdots\cdots & 1 \\ \frac{c_m}{\omega^m} & \frac{c_{m-1}}{\omega^{m-1}} & \cdots\cdots & \frac{c_1}{\omega} \end{bmatrix} \equiv [M_{ij}]$$

Then (12) can be written in the form

$$(15)\quad \frac{d\,\underline{W}_0(\omega,y)}{dy} = \omega\,\mathcal{M}(\omega,y)\,\underline{W}_0(\omega,y)\ ,\quad \underline{W}_0(\omega,0) = \tilde{\underline{V}}(\omega)$$

or equivalently $\underline{W}_0(\omega,y)$ can be expressed as the product

$$(16)\quad \underline{W}_0(\omega,y) = \mathcal{W}(\omega,y)\,\tilde{\underline{V}}(\omega)\ ,$$

where the matrix $\mathcal{W}(\omega,y)$ satisfies the matrix differential equation

$$(17)\quad \frac{d\,\mathcal{W}(\omega,y)}{dy} = \omega\,\mathcal{M}(\omega,y)\,\mathcal{W}(\omega,y)$$

and $\mathcal{W}(\omega,0) = \mathcal{I}$ , the identity matrix of order m.

Since $c_k$ is a polynomial of degree k in $\omega$ and is bounded in y for $0 \leq y \leq h$, we have that $\mathcal{M}$ is bounded uniformly for $|\omega| > 1$ and $0 \leq y \leq h$ . Call this bound $\mu$. An upper bound for $\mu$ in terms of the coefficients $a_{pq}$ can be obtained explicitly:*

$$\begin{aligned} ||\mathcal{M}(\omega,y)||^2 &\equiv \sum_{i,j=1}^{m} |M_{ij}(\omega,y)|^2 \leq (m-1)^2 + \sum_{j=1}^{m} \left|\frac{c_j}{\omega^j}\right|^2 \\ &= (m-1)^2 + \sum_{j=1}^{m} \omega^{-2j} \left|\sum_{p=o}^{i} a_{p,m-j}(y)(i\omega)^p\right|^2 \\ &\leq (m-1)^2 + \sum_{j=1}^{m} \left[\sum_{p=o}^{j} |\bar{a}_{p,m-j}|\right]^2 \end{aligned}$$

---

*We use the same symbol || || for both the matrix and $L_2$ norms. It will always be clear from the context which is meant.

where $\bar{a}_{p,m-j} = \max\limits_{0\leq y\leq h} a_{p,m-j}(y)$. Then from (17) and repeated application of the Cauchy inequality it follows that

$$(18)\qquad \|\widetilde{\mathcal{W}}(\omega,y)\| \leq \|\mathcal{J}\| e^{\mu y|\omega|} = \sqrt{m}\, e^{\mu y|\omega|} \quad ,$$

so that from (16)

$$(18a)\qquad \|\underline{U}_0(\omega,y)\| \leq \|\widetilde{\mathcal{W}}(\omega,y)\|\cdot\|\tilde{\underline{V}}(\omega)\|$$

$$\leq \sqrt{m}\, e^{\mu y|\omega|}\left\{\sum_{q=0}^{m-1} \omega^{-2q}\, |V_q(\omega)|^2\right\}^{\frac{1}{2}} \quad .$$

In particular then for $|\omega| > 1$

$$(19)\qquad |U_0(\omega,y)| \leq \sqrt{m}\, e^{\mu y|\omega|}\left\{\sum_{q=0}^{m-1} \omega^{-2q}|V_q(\omega)|^2\right\}^{\frac{1}{2}}$$

or uniformly for $0 \leq y \leq h$

$$(20)\qquad |U_0(\omega,y)| \leq \sqrt{m}\, e^{\mu h|\omega|}\left\{\sum_{q=0}^{m-1} \omega^{-2q}|V_q(\omega)|^2\right\}^{\frac{1}{2}}$$

Case B: $|\omega| \leq 1$; here we write (12) as a system in a slightly different way. Define the vectors $\underline{U}_0(\omega,y)$, $\underline{V}(\omega)$ and the matrix $\mathcal{M}(\omega,y)$ by

$$(21)\qquad \underline{U}_0(\omega,y) = \begin{bmatrix} U_0(\omega,y) \\ \dfrac{\partial U_0(\omega,y)}{\partial y} \\ \vdots \\ \dfrac{\partial^{m-1}U_0(\omega,y)}{\partial y^{m-1}} \end{bmatrix}, \quad \underline{V}(\omega) = \begin{bmatrix} V_0(\omega) \\ V_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_{m-1}(\omega) \end{bmatrix}$$

$$(22)\quad \mathcal{N}(\omega,y) = \begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & \cdots\cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots\cdots & 1 \\ c_m & c_{m-1} & & \cdots\cdots & c_1 \end{bmatrix}$$

Then (12) takes the form

$$(23)\quad \frac{d\underline{U_0}(\omega,y)}{dy} = \mathcal{N}(\omega,y)\,\underline{U_0}(\omega,y) \quad , \quad \underline{U_0}(\omega,0) = \underline{V}(\omega)$$

or equivalently

$$(24)\quad \underline{U_0}(\omega,y) = \mathcal{U}(\omega,y)\,\underline{V}(\omega)$$

where the matrix $\mathcal{U}(\omega,y)$ satisfies the differential equation

$$(25)\quad \frac{d\,\mathcal{U}(\omega,y)}{dy} = \mathcal{N}(\omega,y)\,\mathcal{U}(\omega,y)\ , \quad \mathcal{U}(\omega,0) = \mathcal{I}\ .$$

Since $|\omega| \leq 1$, the matrix $\mathcal{N}$ is uniformly bounded for $0 \leq y \leq h$. In fact it is easy to show that $\mathcal{M}$ and $\mathcal{N}$ have the same bound. Thus proceeding as before we find

$$(26)\quad ||\underline{U_0}(\omega,y)|| \leq \sqrt{m}\, e^{\mu y} \left\{ \sum_{q=o}^{m-1} |v_q(\omega)|^2 \right\}^{\frac{1}{2}}$$

or in particular uniformly for $0 \leq y \leq h$

$$(27)\quad |U_0(\omega,y)| \leq \sqrt{m}\, e^{\mu h} \left\{ \sum_{q=o}^{m-1} |v_q(\omega)|^2 \right\}^{\frac{1}{2}} .$$

Recalling the definition (1) for $\phi_k^{\alpha}(\omega)$ we can combine the inequalities (20) and (27) by defining

$$(28)\quad \alpha = \mu h$$

and writing

$$(29)\quad |U_0(\omega,y)| \le \|\underline{U_0}(\omega,y)\| \le \sqrt{m}\left\{\sum_{q=0}^{m-1} |\phi^{\alpha}_{-q}(\omega)V_q(\omega)|^2\right\}^{\frac{1}{2}} .$$

The solution $U_0(\omega,y)$ to the homogeneous equation (12) is given either by the first component of (16) which is actually valid for all y and any $\omega$-region excluding a neighborhood of $\omega = 0$, or by (24) which is valid for all y and $\omega$. The purpose for the formal separation into the Cases A and B is to obtain the estimate (29) which is sharper than would result from a consideration of the system (24), (25) alone. A similar procedure will be employed in considering the solution $U(\omega,y)$ to the nonhomogeneous problem (8), (11).

Define the vectors $\underline{U}(\omega,y)$, $\underline{W}(\omega,y)$ and $[\underline{F_u(\omega,y)}]_\beta$ by

$$(30)\quad \underline{W}(\omega,y) = \begin{bmatrix} U(\omega,y) \\ \frac{1}{\omega}\frac{\partial U(\omega,y)}{\partial y} \\ \vdots \\ \frac{1}{\omega^{m-1}}\frac{\partial^{m-1}U(\omega,y)}{\partial y^{m-1}} \end{bmatrix},$$

$$\underline{U}(\omega,y) = \begin{bmatrix} U(\omega,y) \\ \frac{\partial U(\omega,y)}{\partial y} \\ \vdots \\ \frac{\partial^{m-1}U(\omega,y)}{\partial y^{m-1}} \end{bmatrix}, \quad [\underline{F_u(\omega,y)}]_\beta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ [F_u(\omega,y)]_\beta \end{bmatrix}$$

Then (8) may be written as a system in the following way

$$(31)\quad \frac{d\underline{W}(\omega,y)}{dy} = \omega\,\mathcal{M}(\omega,y)\,\underline{W}(\omega,y)+[\underline{F_u(\omega,y)}]_\beta,\quad \underline{W}(\omega,0)=\underline{\tilde{V}}(\omega);\qquad |\omega| > 1$$

$$(32)\quad \frac{d\underline{U}(\omega,y)}{dy} = \mathcal{N}(\omega,y)\,\underline{U}(\omega,y)+[\underline{F_u(\omega,y)}]_\beta,\quad \underline{U}(\omega,0) = \underline{V}(\omega);\qquad |\omega| \leq 1 .$$

Consider the integral equations

$$(33)\quad \underline{W}(\omega,y) = \underline{W_0}(\omega,y) + \int_0^y \mathcal{W}(\omega,y)\,\mathcal{W}^{-1}(\omega,\ )[\underline{F_u(\omega,\eta)}]_\beta d\eta$$

$$(34)\quad \underline{U}(\omega,y) = \underline{U_0}(\omega,y) + \int_0^y \mathcal{U}(\omega,y)\,\mathcal{U}^{-1}(\omega,\eta)[\underline{F_u(\omega,\eta)}]_\beta d\eta ,$$

where $\underline{U_0}$ and $\underline{W_0}$ are defined as before. From the classical theory of ordinary differential equations $\underline{U_0}$, $\underline{W_0}$, $\mathcal{U}$ and $\mathcal{W} \in C^1(y)$, so we may differentiate in (33) and (34). It follows immediately that these expressions satisfy (31) and (32), respectively, including the required initial conditions. As above we note that the representation (33) is valid in any region which excludes a neighborhood of $\omega = 0$, and (34) holds everywhere.

If we denote the elements of the matrices $\mathcal{U}$ and $\mathcal{U}^{-1}$ by $U_{ij}$ and $U^{-1}_{ij}$, respectively, then using (34) the first component $U(\omega,y)$ of $\underline{U}$ may be expressed in the form

$$(35)\quad U(\omega,y) = U_0(\omega,y) + \int_0^y \sum_{j=1}^{m} U_{1j}(\omega,y)\,U^{-1}_{jm}(\omega,\eta)[F_u(\omega,\eta)]_\beta \cdot d\eta$$

Now let

$$u_0(x,y) = \int_{-\infty}^{\infty} U_0(\omega,y)\,e^{i\omega x}\,d\omega$$

$$u(x,y) = \int_{-\infty}^{\infty} U(\omega,y)e^{i\omega x}\, d\omega$$

$$\left.\begin{aligned} u_{1j}(x,y) &= \int_{-\infty}^{\infty} U_{1j}(\omega,y)e^{i\omega x}\, d\omega \\ u_{jm}^{-1}(x,y) &= \int_{-\infty}^{\infty} U_{jm}^{-1}(\omega,y)e^{i\omega x}\, d\omega \end{aligned}\right\} j = 1,2,\ldots,m$$

$$[f(u(x,y))]_\beta = \int_{-\infty}^{\infty} [F_u(\omega,\eta)]_\beta e^{i\omega x}\, d\omega \;.$$

Then, assuming that $\beta$ can be chosen such that the integral in (34) (and therefore in (35)) is finite, it follows from (35) and an appeal to Fubini's theorem that $u(x,y)$ satisfies the integral equation

$$(36)\quad u(x,y) = u_0(x,y) + \int_0^y \int_{-\infty}^{\infty} k(x-\xi,y,\eta)[f(u(\xi,\eta))]_\beta d\xi d\eta \;,$$

where $k(x,y,\eta)$ is the sum of convolutions:

$$(37)\qquad k(x,y,\eta) = \frac{1}{(2\pi)^2} \sum_{j=1}^{m} u_{1j}(x,y) * u_{jm}^{-1}(x,\eta) \;.$$

Thus, when $\beta$ is chosen in this manner, $u(x,y)$ defined by (36) satisfies the original equation (3) and the boundary conditions (4), since $U(\omega,y)$ as given by (35) satisfies (8) and (11). We have, therefore, reduced the original problem to proving the existence, uniqueness and continuous dependence of a solution to the pair of integral equations (35), (36), or more generally to the pair (34), (36). This proof is given in the following section.

## IV SOLUTION OF THE INTEGRAL EQUATIONS

We consider the integral equations (34), (36). For $p = 0,1,\ldots$ define the iterates $\underline{U}_p$, $u_p$ by

$$(38)\quad \underline{U_{p+1}}(\omega,y) = \underline{U_0}(\omega,y) + \int_0^y \mathcal{U}(\omega,y)\,\mathcal{U}^{-1}(\omega,\eta)\,\cdot\,\underline{[F_{u_p}(\omega,\eta)]_\beta d\eta}$$

$$(39)\quad u_{p+1}(x,y) = u_0(x,y) + \int_0^y \int_{-\infty}^{\infty} k(x-\xi,y,\eta)[f(u_p(\xi,\eta)]_\beta d\xi d\eta$$

where

$$\underline{U_{p+1}}(\omega,y) = \begin{bmatrix} U_{p+1}(\omega,y) \\ \dfrac{\partial U_{p+1}(\omega,y)}{\partial y} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial^{m-1} U_{p+1}(\omega,y)}{\partial y^{m-1}} \end{bmatrix}$$

$$(40)\quad U_{p+1}(\omega,y) = U_0(\omega,y) + \int_0^y \sum_{j=1}^{m} U_{1j}(\omega,y)U^{-1}_{jm}(\omega,\eta)\,\cdot\,[F_{u_p}(\omega,\eta)]_\beta d\eta \quad .$$

In order to obtain sharper estimates in what follows we shall use the representation (38) for $|\omega| \leq 1$ and for $|\omega| > 1$ the equivalent form [*]

$$(41)\quad \underline{W_{p+1}}(\omega,y) = \underline{W_0}(\omega,y) + \int_0^y \mathcal{W}(\omega,y)\,\mathcal{W}^{-1}(\omega,\eta)\,\cdot\,\underline{[F_{u_p}(\omega,\eta)]_\beta}\, d\eta \quad ,$$

---

[*] The expressions(38) and (41) are equivalent for $|\omega| > 1$ (actually in any region including a neighborhood of $\omega = 0$) since $\underline{U_0}$, $\underline{W_0}$, $\mathcal{U}$, $\mathcal{W}$ depend only on the solution to the homogeneous problem.

where

$$\underline{W}_{p+1}(\omega,y) = \begin{bmatrix} U_{p+1}(\omega,y) \\ \frac{1}{\omega}\,\frac{\partial U_{p+1}(\omega,y)}{\partial y} \\ \vdots \\ \frac{1}{\omega^{m-1}}\,\frac{\partial^{m-1} U_{p+1}(\omega,y)}{\partial y^{m-1}} \end{bmatrix} .$$

From the properties of $\underline{U}_0$, $\underline{W}_0$, $\mathcal{U}$ and $\mathcal{W}$ we see that $\underline{U}_{p+1}$ and $\underline{W}_{p+1}$ satisfy the initial conditions (11) and $u_{p+1}$ satisfies (4).

We next obtain bounds for $\|\mathcal{U}(\omega,y)\,\mathcal{U}^{-1}(\omega,\eta)\|$ and $\|\mathcal{W}(\omega,y)\,\mathcal{W}^{-1}(\omega,\eta)\|$.

Case A: Consider the adjoint equation to (17)

$$\frac{d\,\mathcal{W}^*(\omega,y,\eta)}{d\eta} = -\,\mathcal{W}^*(\omega,y,\eta)\,\omega\,\mathfrak{M}(\omega,\eta) \quad , \; 0 \leq \eta \leq y$$

$$\mathcal{W}^*(\omega,y,y) = \mathcal{I} \qquad .$$

It is easy to show that

$$\mathcal{W}^*(\omega,y,\eta) = \mathcal{W}(\omega,\eta)\,\mathcal{W}^{-1}(\omega,\eta) .$$

Therefore, using the results of Section III we have for $|\omega| > 1$ that

$$\|\mathcal{W}(\omega,y)\,\mathcal{W}^{-1}(\omega,\eta)\| \leq \sqrt{m}\, e^{\mu y|\omega|} \qquad .$$

From this inequality and the mean value theorem it follows that

(42) $\|\underline{W}_{p+1}(\omega,y)\| \leq \|\underline{W}_0(\omega,y)\| + y\sqrt{m}\, e^{\alpha|\omega|}\,\|[F_{u_p}(\omega,\zeta(\omega))]_\beta\|$,

where $\zeta$ is chosen appropriately depending on $\omega$, $0 \leq \zeta \leq y \leq h$, and as before $\alpha = \mu h$. But

$$||[F_{u_p}(\omega,\zeta(\omega))]_\beta|| = |[F_{u_p}(\omega,\zeta(\omega))]_\beta|$$

so that combining (42) and (18a) we have

$$(43)\quad ||\underline{W_{p+1}}(\omega,y)|| \le \sqrt{m}\, e^{\alpha|\omega|}\, [\{\sum_{q=o}^{m-1} \omega^{-2q}\, |V_q(\omega)|^2\}^{\frac{1}{2}} + y\, |[F_{u_p}(\omega,\zeta)]_\beta|]$$

Case B: For $|\omega| \le 1$ , we proceed similarly to obtain the estimate

$$||\mathcal{U}(\omega,y)\,\mathcal{U}^{-1}(\omega,\eta)|| < \sqrt{m}\, e^{\mu y}$$

Thus employing the mean value theorem and the results of Section III we obtain from (38) the inequality

$$(44)\quad ||\underline{U_{p+1}}(\omega,y)|| \le \sqrt{m}\, e^{\alpha}\, [\{\sum_{q=o}^{m-1} |V_q(\omega)|^2\}^{\frac{1}{2}} + y|[F_{u_p}(\omega,\zeta)]_\beta|]$$

We may combine (43) and (44) by using (29). One obtains for the first component $U_{p+1}(\omega,y)$ defined by (40)

$$(45)\quad |U_{p+1}(\omega,y)| \le \sqrt{m}\, [\{\sum_{q=o}^{m-1} |\phi^{\alpha}_{-q}(\omega)V_q(\omega)|^2\}^{\frac{1}{2}} + y e^{\alpha\Omega(\omega)} \cdot |[F_{u_p}(\omega,\zeta)]_\beta],$$

where

$$\Omega(\omega) = \max\{1, |\omega|\} .$$

We have assumed that $v_q(x) \in E^{\alpha}_{m-q}(x)$ with $\alpha$ given by (28). Thus $\phi^{\alpha}_{m-q}V_q \in L_2$ for $q = 0,1,\dots,m-1$ so certainly $\phi^{\alpha}_{-q}V_q \in L_2$ for all such q. Let*

$$(46)\qquad C = \sum_{q=o}^{m-1} ||\phi^{\alpha}_{-q}V_q|| .$$

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*Here and in what follows the double bar denotes the $L_2$ norm .

We now choose $\beta$ to be a real constant satisfying $\beta > \alpha$; say $\beta = \alpha + \delta$, where $\delta$ is positive but otherwise arbitrary. Then from the definition of the operator $[\quad]_\beta$, the inequality (7) and the Plancherel theorem we have

$$|[F_{u_p}(\omega,\zeta)]_\beta| < A_\lambda \, ||U_p(y)||^2 e^{-\beta|\omega|} .$$

Therefore

$$||e^{\alpha\Omega}|[F_{u_p}(\zeta)]_\beta||^2 = \int_{|\omega|\leq 1} e^{2\alpha\Omega(\omega)}|F_{u_p}(\omega,\zeta)|^2 d\omega + $$

$$+ \int_{|\omega|<1} e^{2\alpha\Omega(\omega)}|F_{u_p}(\omega,\zeta)|^2 d\omega$$

$$(47) \qquad \leq A_\lambda^2 \, ||U_p(y)||^4 \left\{ \int_{|\omega|\leq 1} e^{-2\delta|\omega|} \, d\omega + \int_{|\omega|>1} e^{2(\alpha-\beta|\omega|)} d\omega \right\}$$

$$= A_\lambda^2 \, ||U_p(y)||^4 \left\{ \frac{1}{\beta} \left(\frac{\alpha}{\delta} e^{-2\delta} + e^{2\alpha}\right) \right\} .$$

Combining (45), (46) and (47) we obtain

$$||U_{p+1}(y)|| \leq \sqrt{m}\, C + y\sqrt{m/\beta}\, A_\lambda \left( \frac{\alpha}{\delta} e^{-2\delta} + e^{2\alpha}\right)^{\frac{1}{2}} ||U_p(y)||^2,$$

or uniformly for $y \in [0,h]$

$$(48) \quad ||U_{p+1}|| \leq \sqrt{m}\, C + h\sqrt{m/\beta}\, A_\lambda\left( \frac{\alpha}{\delta} e^{-2\delta} + e^{2\alpha}\right)^{\frac{1}{2}} ||U_p||^2 ,$$

where

$$||U_p|| = \max_{0\leq y\leq h} ||U_p(y)|| .$$

Now if $||U_p|| < 2C\sqrt{m}$, then from (48) it follows that $||U_{p+1}|| < 2C\sqrt{m}$ as long as

$$(49)\quad A_\lambda h\beta^{-\frac{1}{2}} \left(\frac{\alpha}{\delta} e^{-2\delta} + e^{2\alpha}\right) 4C < 1 .$$

Thus we have inductively that $||U_p|| < 2C\sqrt{m}$ for all p under the condition (49). To see more clearly what this condition implies for h, we write $\delta = h\nu$ (recall $\alpha = h\mu$) so that $\beta = h(\mu+\nu)$. Thus if we explicitly define the sphere $S_\lambda$, by setting $\lambda = \sqrt{8\pi m}\, C$ we have from the above results and the Plancherel theorem that the iterates $u_p$ defined by (39) all lie in $S_\lambda$ as long as h is sufficiently small for

$$(50)\quad \sqrt{h}\ \left(\frac{\mu}{\nu} e^{-2h\nu} + e^{2h\mu}\right) < [4CA_\lambda]^{-1} \sqrt{\mu+\nu}$$

to hold[*].

We now estimate the difference $||U_{p+1} - U_p||$ and show that the sequence $\{U_p\}$ converges in $L_2$. Proceeding as before it follows that

$$|U_{p+1}(\omega,y) - U_p(\omega,y)| \le y\sqrt{m}\, e^{\alpha\Omega(\omega)} |[F_{u_p}(\omega,\zeta)]_\beta - [F_{u_{p-1}}(\omega,\zeta)]_\beta|$$

$$(51)\qquad = y\sqrt{m}\, e^{\alpha\Omega(\omega)-\beta|\omega|} |F_{u_p}(\omega,\zeta) - F_{u_{p-1}}(\omega,\zeta)| \quad .$$

Now $\alpha\Omega - \beta|\omega| = -\delta|\omega|$ if $|\omega| > 1$ and $\alpha - \beta|\omega| < \alpha$ if $|\omega| \le 1$. In any event $e^{\alpha\Omega - \beta|\omega|} \le e^{\alpha}$ so that from (51)

$$||U_{p+1}(y) - U_p(y)|| \le h\sqrt{m}\, e^{\alpha}\ ||F_{u_p}(\zeta) - F_{u_{p-1}}(\zeta)|| \quad .$$

Thus applying the Lipschitz condition on f and the Plancherel theorem we obtain

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[*] Note that as $\nu \to \infty$ (in which case (3) degenerates into the linear problem $Lu \equiv 0$) the solution given by our method reduces, as it should, to the unique global one guaranteed by the classical theory.

$$\begin{aligned} ||U_{p+1}(y) - U_p(y)|| &\leq h\sqrt{m/2\pi}\, e^{\alpha}\, ||f(u_p(\zeta)) - f(u_{p-1}(\zeta))|| \\ &\leq h\sqrt{m/2\pi}\, e^{\alpha} K_\lambda\, ||u_p(\zeta) - u_{p-1}(\zeta)|| \\ &= h\sqrt{m}\, e^{\alpha} K_\lambda\, ||U_p(\zeta) - U_{p-1}(\zeta)|| \\ &\leq h\sqrt{m}\, e^{\alpha} K_\lambda\, ||U_p - U_{p-1}|| \end{aligned}$$

for all $u_p \in S_\lambda$, $\lambda = \sqrt{8\pi m}\, C$, and, consequently,

$$||U_{p+1} - U_p|| \leq h\sqrt{m}\, e^{\alpha} K_\lambda\, ||U_p - U_{p-1}||.$$

Therefore if (50) holds and in addition

$$(52) \qquad h\, e^{h\mu} < [\sqrt{m}\, K_\lambda]^{-1},$$

the sequence $\{U_p\}$ defined by (38) is a Cauchy sequence in $L_2$ so that the limit function

$$U(\omega, y) = \lim_{p \to \infty} U_p(\omega, y)$$

(for each y)

exists belongs to $L_2$ and satisfies the integral equation (35). Similarly from the equivalence of (39) and (40), and the uniqueness of the Fourier transform, the limit of the iterates defined by (39)

$$u(x,y) = \lim_{p \to \infty} u_p(x,y)$$

exists, belongs to $S_\lambda$ ($\subset L_2$) and satisfies (36). This completes the proof of existence of the solutions to the integral equation (and thus to the original problem (3), (4)) in the region R(h), where h satisfies (50) and (52). There remains to show uniqueness, differentiability ($C^m(x,y)$) and continuous dependence of the solution.

The uniqueness of the solution $U_0(\omega,y)$ to the homogeneous equation (12) follows from the classical theory of ordinary differential equations. Thus, to prove the uniqueness of the solution pair $U(\omega,y)$, $u(x,y)$ it is sufficient to consider a second solution $U'(\omega,y)$, $u'(x,y)$ to the integral equations

$$\underline{U}'(\omega,y) = \underline{U_0}(\omega,y) + \int_0^y \mathcal{U}(\omega,y)\ \mathcal{U}^{-1}(\omega,\eta)\ \cdot$$

(53)

$$\cdot\ [F_{u'}(\omega,\eta)]_\beta d\eta$$

$$u'(x,y) = u_0(x,y) + \int_0^y \int_{-\infty}^{\infty} k(x-\xi,y,\eta)[f(u'(\xi,\eta))]_\beta d\xi d\eta \quad .$$

Proceeding as above we find from (38) and (53)

$$||U'(y) - U_p(y)|| \leq h\sqrt{m}\ K_\lambda e^a\ ||U'(y) - U_{p-1}(y)||$$

from which it follows

$$||U' - U_p|| \to 0$$

whenever (50) and (52) are satisfied. Thus from the uniqueness of the Fourier transform

$$||u' - u_p|| \to 0$$

so that the solution $u(x,y)$ is unique in $L_2(-\infty,+\infty)$, $y \in [0,h]$, $h$ satisfying (50) and (52).

It follows from the classical theory that $\underline{U_0}(\omega,y)$ and $\mathcal{U}(\omega,y) \in C^1(y)$. Thus, since these quantities involve derivatives of $U_0(\omega,y)$ of order m-1 in y, we have from (35) (or (33)) that $U(\omega,y) \subset C^m(y)$ and, consequently, $u(x,y) \in C^m(y)$ for

$0 \leq y \leq h$. To prove $u(x,y) \in C^m(x,y)$ it is therefore sufficient to show that

$$(54) \qquad \omega^m U(\omega,y) \in L_2 \ .$$

From the definition (1) and (45) we obtain

$$|\omega^m U(\omega,y)| \leq \sqrt{m} \left\{ \sum_{q=o}^{m-1} |\phi^{\alpha}_{m-q}(\omega) V_q(\omega)|^2 \right\}^{\frac{1}{2}}$$

$$(55) \qquad + y\sqrt{m}\, e^{\alpha\Omega(\omega)} |[F_u(\omega,\zeta)]_\beta| |\omega^m|$$

Since $v_q(x) \in E^{\alpha}_{m-q}(x)$ by assumption, the first term in (55) belongs to $L_2$ and (54) will therefore follow if we can show

$$(56) \qquad e^{\alpha\Omega(\omega)} \omega^m |[F_u(\omega,\zeta)]_\beta| \in L_2$$

for all $\zeta \in [0,h]$. But

$$\omega^m e^{\alpha\Omega(\omega)} [F_u(\omega,\zeta)]_\beta \leq A_\lambda ||U||^2 \ |\omega|^m e^{\alpha\Omega - \beta|\omega|}$$

and thus

$$||\omega^m e^{\alpha\Omega} [F_u(\zeta)]_\beta||^2 \leq A_\lambda^2 ||U||^4 \left\{ \int_{|\omega|>1} \omega^{2m} e^{-2\delta|\omega|} d\omega + \int_{|\omega|\leq 1} \omega^{2m} e^{2(\alpha-\beta|\omega|)} d\omega \right\}$$

uniformly for $\zeta \in [0,h]$. Since the integrals are finite for $\delta > 0$, the desired result (56) follows.

We next consider the continuous dependence of the solution. Let $U(\omega,y)$, $u(x,y)$ and $\overline{U}(\omega,y)$, $\overline{u}(x,y)$ be solutions to (8) and (3), corresponding to the initial data $\{V_q(\omega)\}$, $\{v_q(x)\}$ and $\{\overline{V}_q(\omega)\}$, $\{\overline{v}_q(x)\}$, respectively. We assume $v_q(x)$, $\overline{v}_q(x) \in E^{\alpha}_{m-q}(x)$ for $q = 0,1,\ldots,m-1$, where $\alpha$ (which depends only on the

operator L) is given by (28). Thus

$$V_q(\omega)\ \phi^{\alpha}_{m-q}(\omega) \in L_2$$

$$\overline{V}_q(\omega)\ \phi^{\alpha}_{m-q}(\omega) \in L_2 \quad ,$$

$q = 0,1,\ldots,m-1$. Suppose the data $v_q$ and $\overline{v}_q$ are close in the sense that

$$(57) \qquad ||\phi^{\alpha}_{m-q}(V_q - \overline{V}_q)|| < \varepsilon$$

for $q = 0,1,\ldots,m-1$, and some $\varepsilon > 0$. Then certainly

$$(58) \qquad ||\phi^{\alpha}_{-q}(V_q - \overline{V}_q)|| < \varepsilon$$

for $q = 0,1,\ldots,m-1$. Proceeding in the usual manner we derive the inequality

$$|U(\omega,y) - \overline{U}(\omega,y)| \leq \sqrt{m}\left\{\sum_{q=o}^{m-1} |\phi^{\alpha}_{-q}(\omega)|^2 |V_q(\omega) - \overline{V}_q(\omega)|^2\right\}^{\frac{1}{2}}$$

$$+\ y\sqrt{m}\ e^{\alpha\Omega(\omega)}\ |[F_u(\omega,\zeta)]_\beta - [F_{\overline{u}}(\omega,\zeta)]_\beta| \ .$$

Thus defining the sphere $S_\lambda$ by

$$\lambda = \max\left\{\sqrt{8\pi m}\ \sum ||\phi^{\alpha}_{-q}V_q||,\ \sqrt{8\pi m}\ \sum ||\phi^{\alpha}_{-q}\overline{V}_q||\right\}$$

and assuming that condition (iii) on $f(u)$ holds for $u \in S_\lambda$, we have

$$||U(y) - \overline{U}(y)|| \leq \sqrt{m}\left\{\sum_{q=o}^{m-1} ||\phi^{\alpha}_{-q}(V_q - \overline{V}_q)||^2\right\}^{\frac{1}{2}} + y\sqrt{m}\ e^{\alpha}\ K_\lambda$$

$$||U(\zeta) - \overline{U}(\zeta)|| \ .$$

Whence from (58)

$$||U - \overline{U}|| \leq m\varepsilon + h\sqrt{m}\, e^{a} K_\lambda ||U-\overline{U}||$$

so that, if (50) and (52) hold,

$$||U-\overline{U}|| \leq \frac{m}{1 - h\sqrt{m}\, e^{a} K_\lambda}\, \varepsilon$$

Thus

$$||u-\overline{u}|| \leq \frac{m\sqrt{2\pi}}{1-h\sqrt{m}\, e^{a} K_\lambda}\, \varepsilon = \frac{m\sqrt{2\pi}}{1-h\sqrt{m}\, e^{h\mu} K_\lambda} \cdot \varepsilon$$

and the solution u to (3) depends Lipschitz continuously on the initial data (4).

Finally we show that the solutions U and u depend continuously on the function f(u). Let $\overline{U}(\omega,y)$ $\overline{u}(x,y)$ be the solutions of the equations $\mathcal{L}\, U = [G_u(\omega,y)]_\beta$ and $Lu = [g(u)]_\beta$, respectively, with the same initial data (11) and (4). Without loss of generality we may suppose that f and g satisfy conditions (i), (ii), (iii) in the sphere $S_\lambda$, $\lambda = \sqrt{8\pi m}\, C$, defined by (46). From condition (ii) we have that $f(u(x,y)), g(u(x,y)) \in L_1$ as functions of x for $u \in S_\lambda$. We suppose that f,g are close in the sense that

$$(59) \qquad \max_{0\leq y\leq h} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(u(x,y)) - g(u(x,y))|dx < \varepsilon \quad .$$

Then using the same estimation procedures as before it follows that

$$||u-\overline{u}|| \leq \frac{\sqrt{2\pi h m}\,(\nu+\mu)^{-\frac{1}{2}}\left(\frac{\mu}{\nu}\, e^{-2h\nu} + e^{2h\mu}\right)^{\frac{1}{2}}}{1-h\sqrt{m}\, K_\lambda e^{h\mu}} \cdot \varepsilon$$

as long as (50) and (52) hold.

We may summarize the results of the above sections in the following

Theorem: Consider the Cauchy problem (3), (4), where the operator L is defined by (2). Let C be given by (46) and the sphere $S_\lambda$ defined by $\lambda = \sqrt{8\pi m}\, C$. Suppose $\mu$ is a bound for the matrices $\mathcal{M}$ and $\mathcal{N}$ given by (14) and (22), respectively.

Then, if f(u) satisfies conditions (i), (ii), (iii) for $u \in S_\lambda$, $v_q(x) \in E^{h\mu}_{m-q}$ $(q = 0,1,\ldots,m-1$ and h satisfying (50), (52)) and $\beta$ is chosen larger than $h\mu$, there exists a solution $u(x,y)$ to (3),(4) in $R(h) = \left\{ (x,y) : -\infty < x < \infty ;\ 0 \leq y \leq h \right\}$ with the following properties

a) $u(x,y) \in E^0_m(x,h)$ in R(h)

b) u(x,y) is unique in the class of functions of x and y which are in $L_2$ with respect to x uniformly in y.

c) u(x,y) depends Lipschitz continuously (in the norm: $\max\limits_{0 \leq y \leq h} \left\{ \int_{-\infty}^{\infty} |u(x,y)|^2 dx \right\}^{\frac{1}{2}}$) on the initial data $v_q(x)$ and the function f(u), when the norms of these quantities are defined by (57) and (59), respectively.

The extension of our methods to the Cauchy problems for the equations $Lu = [f(u, \frac{\partial u}{\partial x})]_\beta$, $Lu = [f(u, \frac{\partial u}{\partial y})]_\beta$ and most generally

$$Lu = [f(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y}, \ldots, \frac{\partial^{m-1} u}{\partial x^{m-1}}, \frac{\partial^{m-1} u}{\partial y^{m-1}}]_\beta ,$$

where L is defined by (2), can be carried out by an adaptation of the methods of Ref [4] in a manner completely analogous to our present treatment of the problem $Lu = [f(u)]_\beta$ .

J.M. Zimmerman
July 2, 1962

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Zimmerman

A class of improper boundary value problems with "damped"

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